J. Ind. Geophys. Union (April 2005)
Vol.9, No.2, pp.127-136

Analytical expression for hydraulic head distribution in a homogeneous anisotropic aquifer with inclined bedding planes

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ABSTRACT
The existing analogy between dc current flow and ground water flow under steady state conditionsin earth medium, has allowed to extend the results from geoelectrical method in computing heads in a homogeneous anisotropic aquifer system with inclined bedding planes due to a surface water source. The results are presented as equipotential (hydraulic head) plots for different coefficients of anisotropy and orientation of bedding planes of soil strata.

INTRODUCTION

Ground water flow problems generally assume homogeneous and isotropic porous media with respect to hydraulic conductivity. However, all layered soil formations exhibit anisotropy due to stratification. Due to anisotropy, the directions of flow and of the hydraulic gradient will not be parallel (Marcus 1962) to each other. Further, several numerical ground water flow models assume that principal axes of anisotropy coincide with the reference coordinate axes (Mc Donald & Harbaugh 1984).

The hydraulic conductivity is influenced by *heterogeneity* and *anisotropy* (Freeze & Cherry 1979), which need to be clearly distinguished. In case of anisotropy the hydraulic conductivity varies along with the direction of measurement only at a given point whereas in case of heterogeneity, it varies throughout space within a geologic formation. The directions in space at which the hydraulic conductivity, K attains its maximum and minimum values are called the *principal directions of anisotropy* and they are always orthogonal to each another (Freeze & Cherry 1979).

The theory of flow of fluids through anisotropic porous medium is presented by Marcus (1962), Scheidegger (1957), Polubarinova-Kochina (1962) and Harr (1962). Some investigations on the transformation of anisotropic medium to isotropic medium are available (Mishra 1972; Strack 1989). Further, theory on geoelectric sounding in homogeneous anisotropic earth medium of inclined bedding planes provides analytical results for the computation of surface distribution of electric potentials due to a point d.c current source (Bhattacharyya & Patra 1968).

In the present effort, by considering the analogy between dc current flow and ground water flow under steady state conditions (Wolfe & Bodl 1997; Pujari 1998), the analytical procedure for computing heads in a homogeneous anisotropic aquifer system with inclined bedding planes due to a surface water source is derived from fundamentals from the existing geophysical literature on d.c current flow in homogeneous anisotropic earth medium. The numerical results are presented as equipotential (hydraulic head) plots for different coefficients of anisotropy and orientation of bedding planes of soil strata.

THEORETICAL ASPECTS

In groundwater problems soil body is considered to be a continuous medium of many , which serve as the fluid carrier. Fluid flow (Harr 1962) in a porous medium with interconnected openings is governed by Darcy's law Viz.,

$$q = - K \frac{\partial \phi}{\partial x} \tag{1}$$

where q [LT^{-1}] is the specific discharge, $\partial\phi/\partial x$ is the hydraulic gradient due to change in hydraulic head ϕ[L] over the distance, x[L], and K[LT^{-1}] is the *hydraulic conductivity*. K is related to the intrinsic permeability of the medium k[L^2], fluid density ρ[ML^{-3}], dynamic viscosity of the fluid μ[$ML^{-1}T^{-1}$], and acceleration due to gravity g[LT^{-2}] by the equation

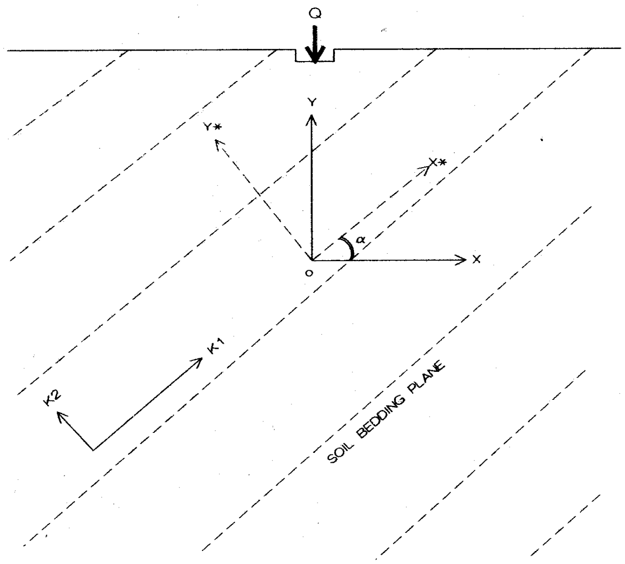

Figure 1. Scheme of the homogeneous anisotropic aquifer system with infinite extent with a point source, Q located centrally on the surface. Dotted lines represent bedding planes of soil strata with angle of dip, α.

$$K = \frac{k\,\rho g}{\mu} \qquad (2)$$

The hydraulic conductivity parallel to the layer, K_1 in the direction of x*-axis (Fig. 1) being larger in magnitude than the one K_2, perpendicular, (in the direction of y*-axis) to the bedding plane. Let α be the angle of dip. Let, also, (x, y) and (x*, y*) be the Cartesian coordinates of an arbitrary point P in the actual plane and the rotated plane respectively.

The plane of stratification of the soil bedding can either be parallel to the ground surface or be inclined with an angle of dip, α. It is assumed that the dimensions of the aquifer system extend to infinity. It is assumed that the major principal direction of anisotropy is along the plane of the soil strata while the other is perpendicular to it. Referring to Figure-1, the following relationship holds good:

$$x = x^* \cos\alpha - y^* \sin\alpha \qquad (3)$$
$$y = x^* \sin\alpha + y^* \cos\alpha$$

Let q_x, q_y and q^*_x, q^*_y be the corresponding specific discharge vectors in the actual and rotated planes. The expressions for q_x and q_y in terms of q^*_x and q^*_y are similar to eqn. (3):

$$q_x = q_x^* \cos\alpha - q_y^* \sin\alpha \qquad (4)$$
$$q_y = q_x^* \sin\alpha + q_y^* \cos\alpha$$

Now, application of Darcy's law in terms of x*, y* coordinate system yields:

$$q_x^* = -K_1 \frac{\partial\phi}{\partial x^*} \qquad (5)$$
$$q_y^* = -K_2 \frac{\partial\phi}{\partial y^*}$$

where K_1 and K_2 are the principal values of the hydraulic conductivity.

Using eqn.(5) in eqn.(4), we get:

$$q_x = -K_1 \frac{\partial\phi}{\partial x^*} \cos\alpha + K_2 \frac{\partial\phi}{\partial y^*} \sin\alpha \qquad (6)$$
$$q_y = -K_1 \frac{\partial\phi}{\partial x^*} \sin\alpha - K_2 \frac{\partial\phi}{\partial y^*} \cos\alpha$$

By the application of chain rule to eqn.(3) yields:

$$\frac{\partial\phi}{\partial x^*} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x^*} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x^*} = \frac{\partial\phi}{\partial x}\cos\alpha + \frac{\partial\phi}{\partial y}\sin\alpha \qquad (7)$$
$$\frac{\partial\phi}{\partial y^*} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial y^*} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial y^*} = \frac{\partial\phi}{\partial x}\sin\alpha + \frac{\partial\phi}{\partial y}\cos\alpha$$

Combining eqn.(6) and eqn.(7), the Darcy's law for anisotropic hydraulic conductivity for two-dimensional flow is obtained as:

$$q_x = -K_{xx}\frac{\partial\phi}{\partial x} - K_{xy}\frac{\partial\phi}{\partial y} \qquad (8)$$
$$q_y = -K_{yx}\frac{\partial\phi}{\partial x} - K_{yy}\frac{\partial\phi}{\partial y}$$

where,

$$K_{xx} = K_1\cos^2\alpha + K_2\sin^2\alpha$$
$$K_{xy} = K_{yx} = (K_1 - K_2)\sin\alpha\,\cos\alpha \qquad (9)$$
$$K_{yy} = K_1\sin^2\alpha + K_2\cos^2\alpha$$

In a similar fashion, for the general case of three-dimensional flow in (x,y,z) coordinate system, it can be shown that Darcy's law takes the form:

$$q_x = -K_{xx}\frac{\partial\phi}{\partial x} - K_{xy}\frac{\partial\phi}{\partial y} - K_{xz}\frac{\partial\phi}{\partial z}$$
$$q_y = -K_{yx}\frac{\partial\phi}{\partial x} - K_{yy}\frac{\partial\phi}{\partial y} - K_{yz}\frac{\partial\phi}{\partial z} \qquad (10)$$
$$q_z = -K_{zx}\frac{\partial\phi}{\partial x} - K_{zy}\frac{\partial\phi}{\partial y} - K_{zz}\frac{\partial\phi}{\partial z}$$

The coefficients K_{ij} (i =x, y, z; j = x, y, z) in eqn. (10) are known as the coefficients of the hydraulic conductivity tensor, represented by:

$$\overline{K} = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{xy} & K_{yx} & K_{xx} \\ K_{yy} & K_{xx} & K_{xx} \end{bmatrix} \qquad (11)$$

Eqn. (11) is a symmetric matrix with the diagonal elements K_{xx}, K_{yy} and K_{zz} and $K_{ij}=K_{ji}$ for i =x, y, z; j = x, y, z. Thus, in an anisotropic medium the hydraulic conductivity tensor is actually characterised by six components.

Further, it is possible to orient the coordinate axes along the principal axes of anisotropy such that $K_{xy} = K_{yz} = K_{zx} = 0$. Then, the resulting governing equation for steady state groundwater flow in a homogeneous anisotropic porous medium reduces to:

$$K_{xx} \frac{\partial^2 \phi}{\partial x^{*2}} + K_{yy} \frac{\partial^2 \phi}{\partial y^{*2}} + K_{zz} \frac{\partial^2 \phi}{\partial z^{*2}} = 0 \tag{12}$$

Choosing a new system of coordinates with,

$$\psi = x^{*}\sqrt{K_{xx}}, \quad \eta = y^{*}\sqrt{K_{yy}}, \quad \zeta = z^{*}\sqrt{K_{zz}} \tag{13}$$

the above equation transforms to the Laplace's form:

$$\frac{\partial^2 \phi}{\partial \psi^2} + \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial^2 \phi}{\partial \zeta^2} = 0 \tag{14}$$

The solution (Bhattacharyya and Patra, 1968) of which is given by:

$$\phi = \frac{C}{\sqrt{\psi^2 + \eta^2 + \zeta^2}} = \frac{C}{\sqrt{\frac{x^{*2}}{K_{xx}} + \frac{x^{*2}}{K_{yy}} + \frac{x^{*2}}{K_{zz}}}} \tag{15}$$

DERIVATION OF HYDRAULIC HEADS

Let us consider the plane of stratification as the XY-plane in the homogeneous anisotropic porous medium. Being very small, in most practical cases, the anisotropy in the plane of stratification can be neglected. Thus, let the longitudinal hydraulic conductivity be $K_{xx}=K_{yy}= K_1$ (parallel to the plane of stratification) and the transverse hydraulic conductivity be $K_{zz}= K_2$ (normal to the plane of stratification). It is possible to define two parameters of the anisotropic medium, the coefficient of anisotropy (â) and mean hydraulic conductivity (K_m), as follows:

$$\beta = \sqrt{\frac{K_1}{K_2}} \text{ and } K_m = \sqrt{K_1 K_2}, \text{ such that } K_m = \frac{K_1}{\beta} = \beta K_2 \tag{16}$$

Using these parameters eqn. (15) can be re-written as:

$$\phi = \frac{C K_1^{1/2}}{(x^{*2} + y^{*2} + \beta^2 x^{*2})^{1/2}} \tag{17}$$

Then, the specific discharge components are obtained by taking the respective derivatives (For details see Annexure I) as:

$$q_x^{*} = \frac{K_1^{3/2} C x^{*}}{(x^{*2} + y^{*2} + \beta^2 z^{*2})^{3/2}} \tag{18}$$

$$q_y^{*} = \frac{K_1^{3/2} C y^{*}}{(x^{*2} + y^{*2} + \beta^2 z^{*2})^{3/2}} \tag{19}$$

$$q_z^{*} = \frac{K_1^{3/2} C z^{*}}{(x^{*2} + y^{*2} + \beta^2 z^{*2})^{3/2}} \tag{20}$$

such that the resultant specific discharge, q is given by,

$$q = \sqrt{q_x^{*2} + q_y^{*2} + q_z^{*2}} = \frac{K_1^{3/2}\, C\, (x^{*2}+y^{*2}+z^{*2})^{1/2}}{(x^{*2} + y^{*2} + \beta^2 z^{*2})^{3/2}} \tag{21}$$

Let the point source on the surface of the homogeneous anisotropic medium be of strength Q (m^3/s). In order to evaluate the constant of integration C, let us consider the total flow through a hemisphere of radius R in the ground beneath the source. Obviously this total outflow should be equal to the total input, Q. Therefore, referring to the spherical coordinate (Bhattacharyya & Patra, 1968) system, we have:

$$Q = \int_s q\, ds = \int_0^{2\Pi} \int_0^{\Pi/2} q\, R^2 \sin\theta\, d\theta\, d\phi \tag{22}$$

Since,

$$x^{*2} + y^{*2} = R^2 \sin^2\theta \tag{23}$$

and

$$z^{*2} = R^2 \cos^2\theta \tag{24}$$

equation (21) becomes

$$q = \frac{K_1^{3/2} C}{(R^2 [1 + (\beta^2 - 1) \cos^2\theta]^{3/2})} \tag{25}$$

and

$$Q = C K_1^{3/2} \int_0^{2\Pi} d\phi \int_0^{\Pi/2} \frac{\sin\theta\, d\theta}{1 + (\beta^2 - 1) \cos^2\theta]^{3/2}} = \frac{2\Pi C K_1^{3/2}}{\beta} \tag{26}$$

Details of integral evaluation in eqn. 26 are provided in Annexure II.

Therefore,

$$C = \frac{Q \beta}{2\Pi K_1^{3/2}} \tag{27}$$

Now, eqn. (17) can be re-written as:

$$\phi = \frac{Q}{2\Pi K_m \sqrt{x^{*2} + y^{*2} + \beta^2 z^{*2}}} \tag{28}$$

The equation (28) provides the hydraulic potential at any point in a horizontally stratified medium due to a point source Q. If the planes of stratification makes an angle of dip α, then the expression for hydraulic potential can be generalised by considering a rotation of the coordinate axes (x,y,z) through an angle α. Let the rotated coordinate axes be (x^*, y^*, z^*). Assuming the strike of the bed x be along x^* the original and the rotated coordinate axes are related by:

$$x^* = x$$
$$y^* = y \cos \alpha + z \sin \alpha \qquad (29)$$
$$z^* = -y \sin \alpha + z \cos \alpha$$

Incorporating the relationship given by eqn. (29) in equation (28) yields the expression for hydraulic potentials in a homogeneous anisotropic medium with inclined planes of stratification as:

$$\phi = \frac{Q}{2\Pi K_m \sqrt{x^2 + (y\cos\alpha + z\sin\alpha)^2 + \beta^2(-y\sin\alpha + z\cos\alpha)^2}} \qquad (30)$$

NUMERICAL EXPERIMENTS

A computational procedure has been devised based on the derived analytical expression (Eqn. 30) for finding the hydraulic head distribution in a homogeneous anisotropic medium. Using input information on source strength (Q), Hydraulic conductivity values in the principal directions (K_1, K_2), angle of dip of the strata (α), and grid (r,z), the proposed method computes the steady state hydraulic heads. The hypothetical aquifer system is formed by a number of strata with inclined bedding planes. The angle of dip (α) of the bedding planes with the horizontal is varied between zero and $\pi/2$ for various cases. The major principal direction of anisotropy is along the bedding plane of the strata and the minor principal direction of anisotropy is perpendicular to it. A point source of strength, Q [L^3T^{-1}] is located at the centre of the system. The boundaries of the hypothetical aquifer system are assumed to be at very large distances from the source thereby extending the aquifer system to infinite distance. Simulation of hydraulic heads are carried out for various levels of anisotropy in the aquifer and different orientations of the strata with appropriate aquifer parameter values.

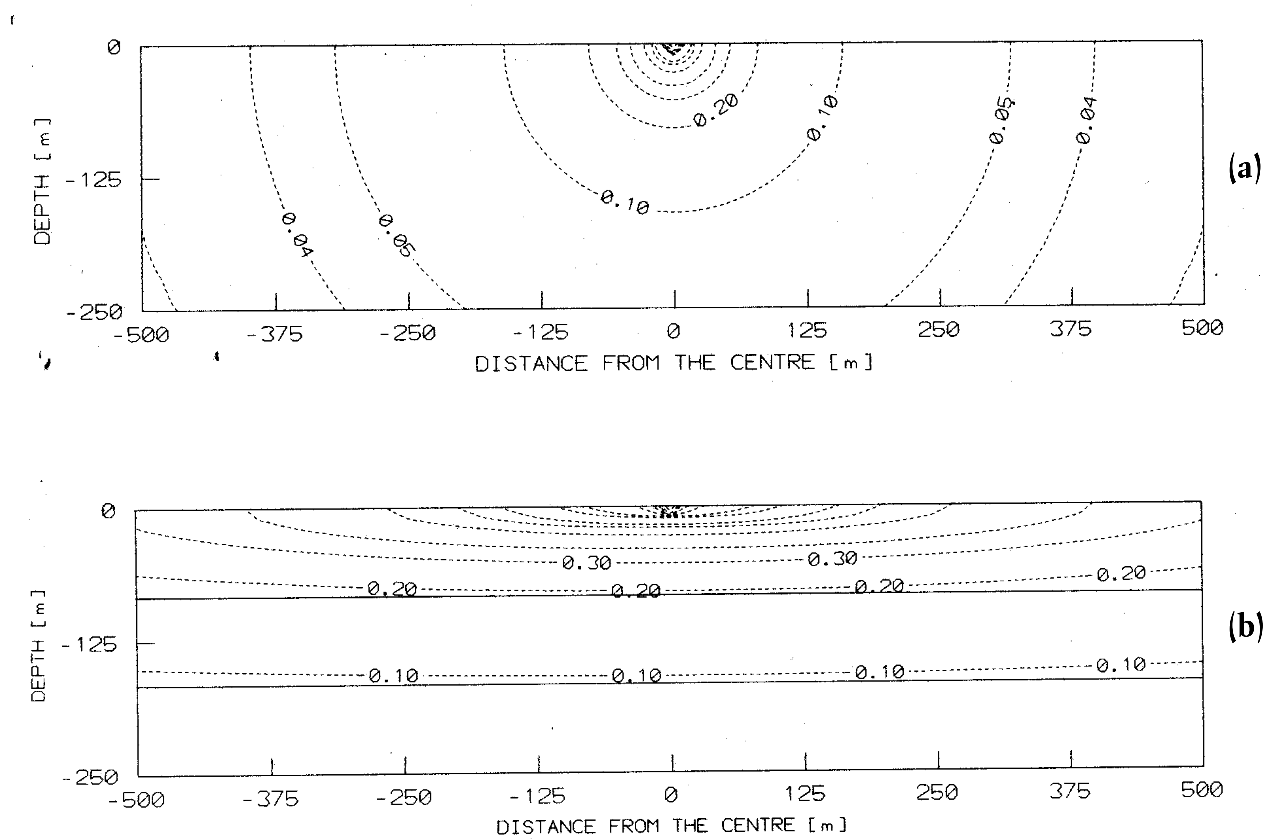

Figure 2. Selected equipotential lines in: [a] an isotropic aquifer where inclination of bedding planes, $\alpha = 0$ and coefficient of anisotropy, $\beta = 1$; [b] in an anisotropic aquifer where inclination of bedding planes, $\alpha = 0$ and coefficient of anisotropy, $\beta = 10$

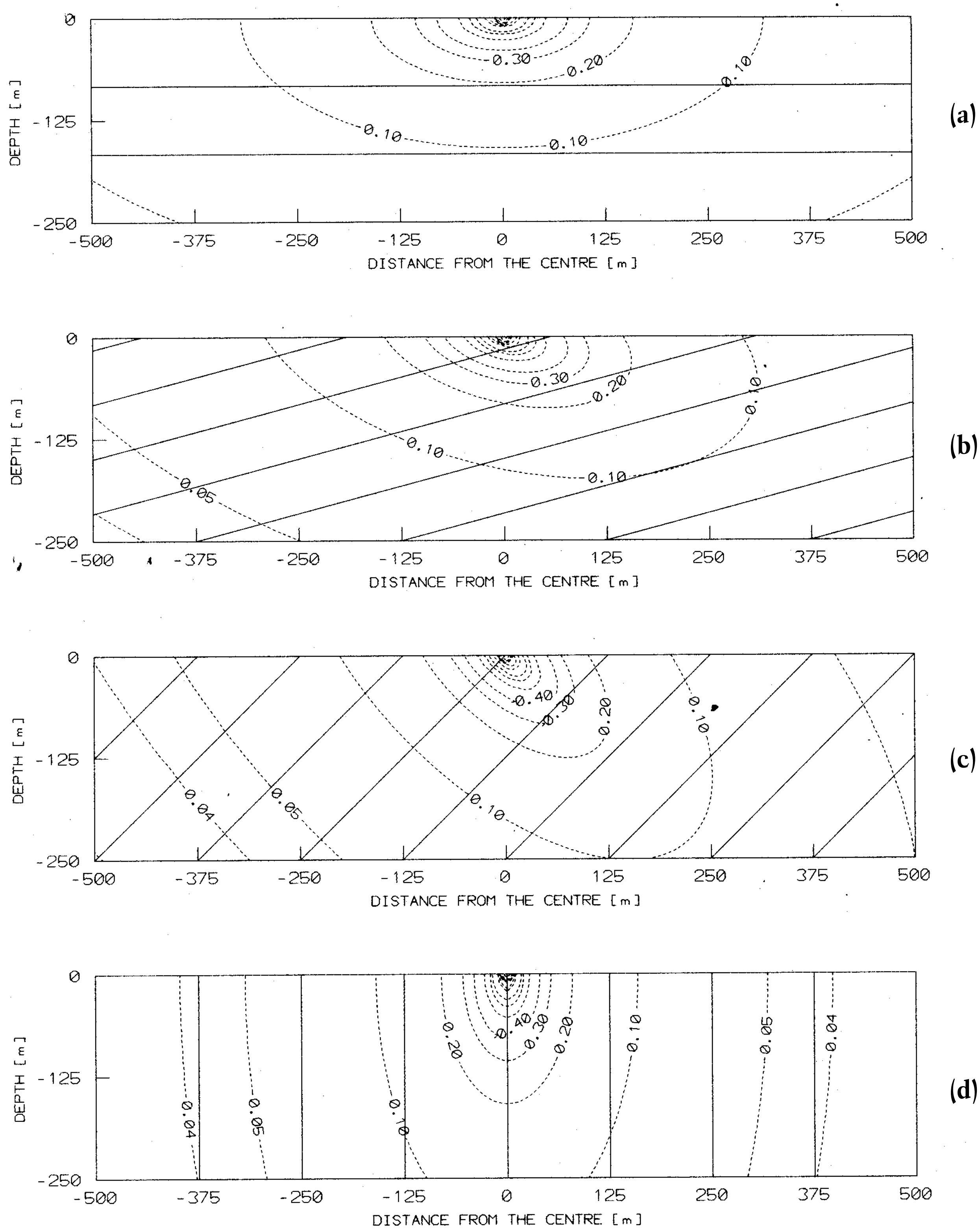

Figure 3. Equi-potentials (dotted lines) in a stratified anisotropic aquifer system for different inclinations (α) of the bedding planes (solid lines) when the coefficient of anisotropy, β=2. [a] For α=0; [b] For α=π/12; [c] For α=π/4; [d] For α=π/2.

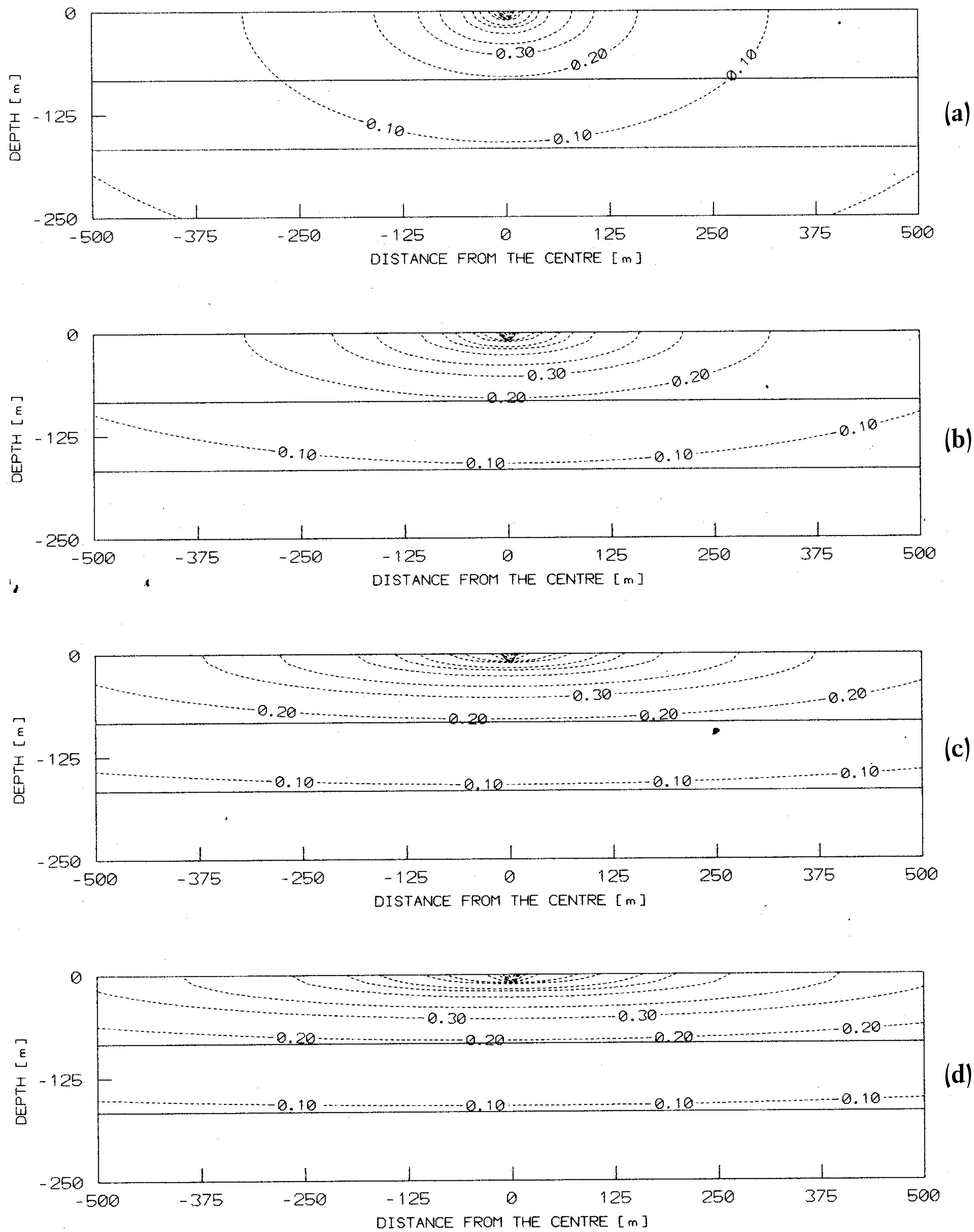

Figure 4. Hydraulic potentials in an anisotropic aquifer for different coefficients of anisotropy (β) when the angle of dip of the strata, α=0. [a] For β=2; [b] For β=4; [c] For β=7; [d] For β=10.

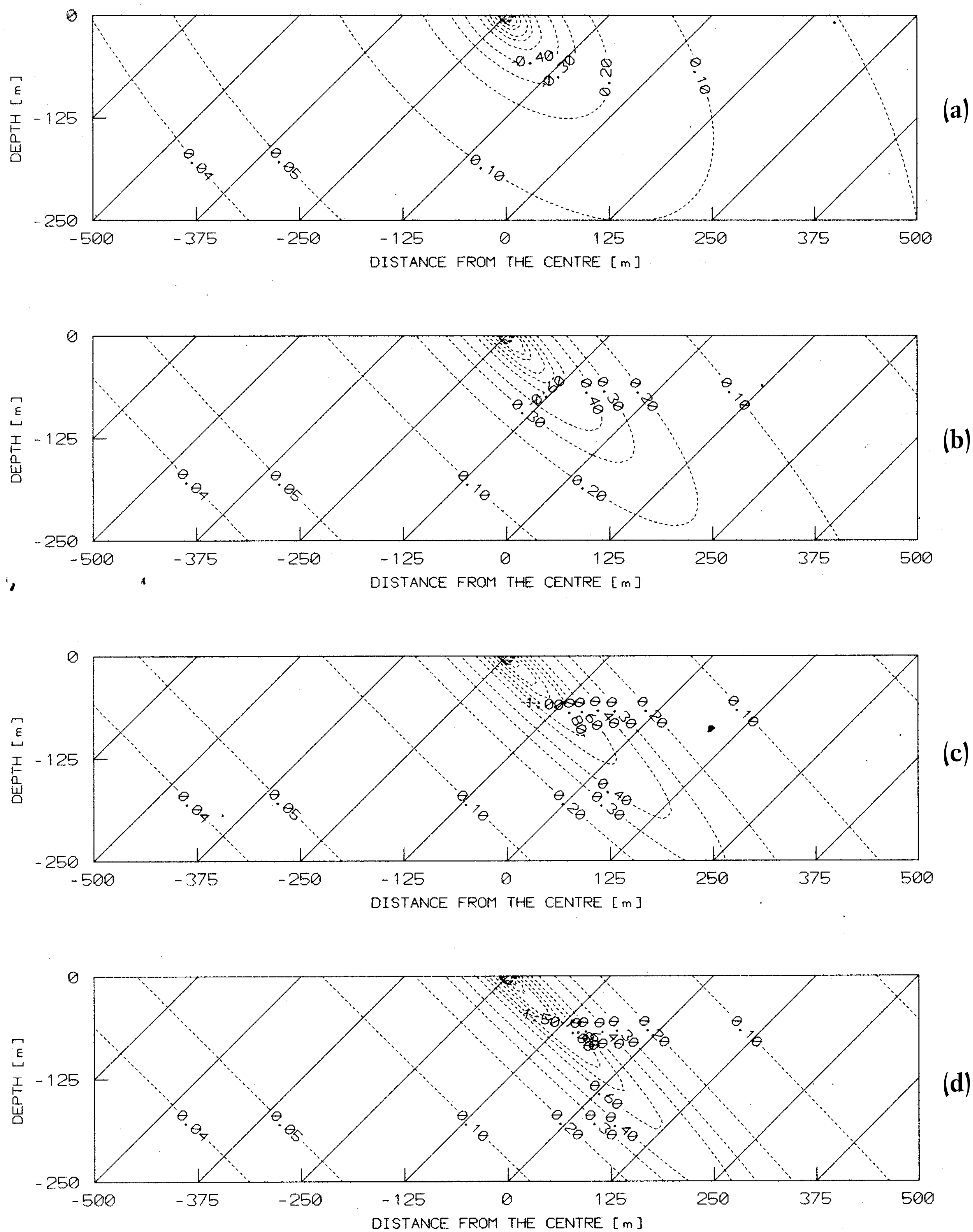

Figure 5. Hydraulic potentials in an anisotropic aquifer for different coefficients of anisotropy (β) when the angle of dip of the strata, $\alpha=\pi/4$. [a] For $\beta=2$; [b] For $\beta=4$; [c] For $\beta=7$; [d] For $\beta=10$.

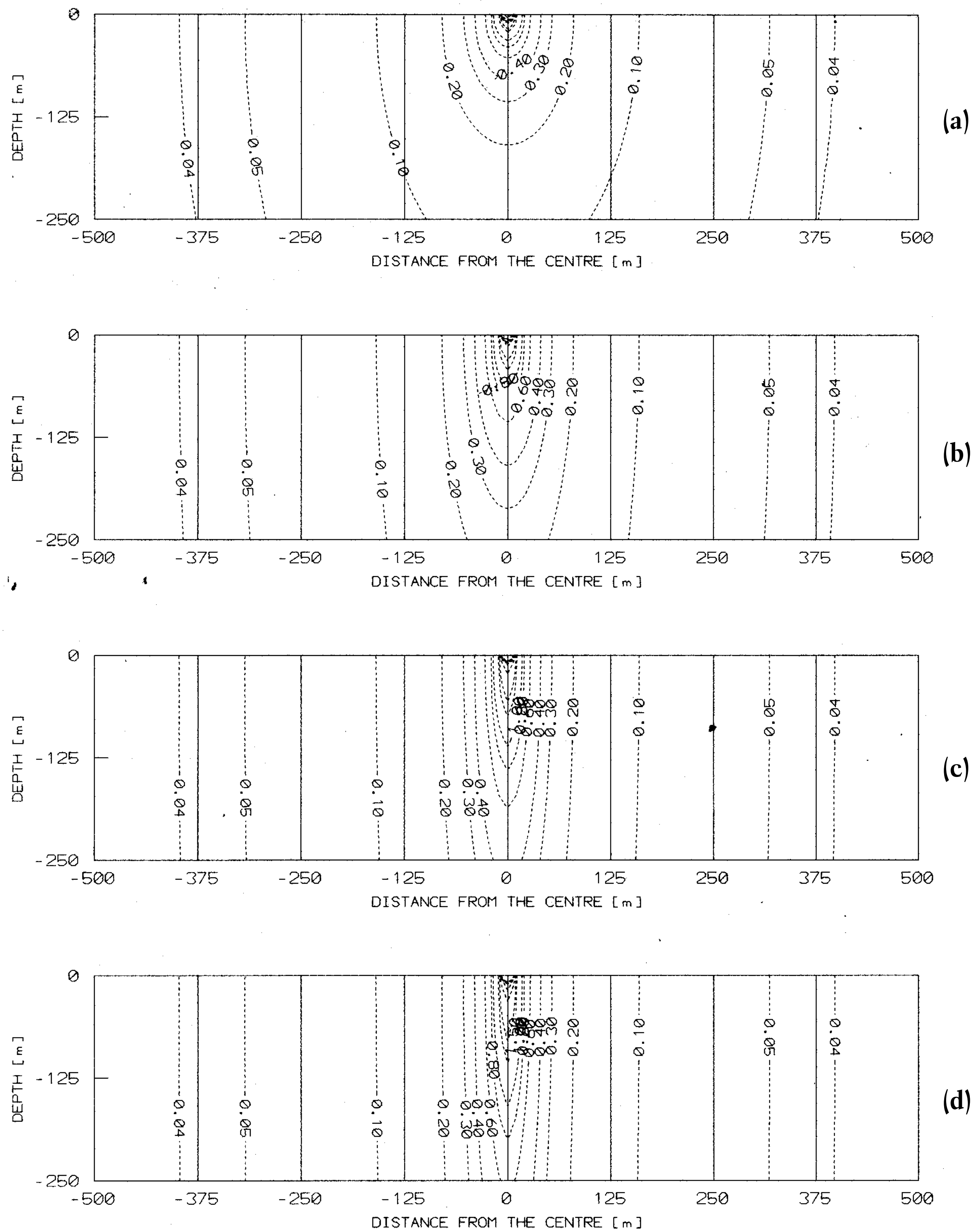

Figure 6. Hydraulic potentials in an anisotropic aquifer for different coefficients of anisotropy (β) when the angle of dip of the strata, $\alpha=\pi/2$. [a] For $\beta=2$; [b] For $\beta=4$; [c] For $\beta=7$; [d] For $\beta=10$.

Fig.2 compares the hydraulic head distribution in an isotropic aquifer with that in an anisotropic aquifer where there is a horizontal stratification. The solid lines inside the plot indicate the stratification. The horizontal hydraulic conductivity in both the cases is K_1=0.001 m/s. In the case of isotropic aquifer (Fig.2a), as K_1 and K_2 are equal, the equipotentials form semi-circles around the source and radial-flow will be taking place uniformly in all directions. However, the hydraulic conductivity K_2, orthogonal to the horizontal hydraulic conductivity, is two orders of magnitude smaller in the case of the anisotropic medium. The shape of equipotentials in the anisotropic case (Fig.2b) is semi-elliptical clearly indicating the tendency of the flow to take place in the least resistive direction.

Fig.3 depicts comparison of the equipotential plots in the anisotropic aquifer system with different orientations of the strata. The thick lines in the plot indicate the orientation of the strata. The hydraulic conductivity values in the principal directions and coefficient of anisotropy, β=2 are the same for all these plots. The ratio of semimajor axis to that of semi-minor axis, a/b is given by denominator of eqn. 30. So, using that expression, one can estimate dip of strata, if cofficient of anisotropy is assumed or vice versa. Here, the effect of dip of strata on equipotential (or hydraulic head distribution) distribution for a given β is clearly seen (Fig. 3a-3d). Indirectly, this helps us visualize that groundwater flow pattern doesn't follow bed dips..

Fig.4 shows the equipotentials in the anisotropic aquifer system with different coefficients of anisotropy. The bedding planes of the strata are horizontal as indicated by the solid lines in the plot. The coefficient of anisotropy applied are β=2, β=4, β=7, and β=10 respectively for the cases (a), (b), (c), and (d) in Figure-4. The equipotentials flatten as the degree of anisotropy becomes larger in the system. Similar kinds of plots as that of the previous one are presented in Figure-5, and Figure-6 with different orientations of the strata. The dips of the soil bedding are taken as $\alpha=\pi/4$, and $\alpha=\pi/2$ respectively for Fig.5 and Fig.6.

The above figures represent selected cases from a spectrum of possible combinations of coefficient of anisotropy and orientation of bedding planes of soil strata in a homogeneous anisotropic aquifer system and enables to visualise the pattern of distribution of hydraulic heads in various cases.

CONCLUSIONS

A analytical expression for hydraulic potential (or head) distribution in homegeneous anisotropic aquifer is developed from a similar result in d.c resistivity method due to point d.c current source in view of existing analogy between d.c current flow and ground water flow under steady state conditions. Hydraulic heads are computed in a homogeneous anisotropic aquifer system with different coefficients of anisotropy, and orientations of soil strata. The study demonstrates usefulness of analytical solution in simulating hydraulic heads in a homogeneous anisotropic aquifer.

ACKNOWLEDGEMENT

The authors acknowledge Prof. G.C. Mishra, Professor, WRDTC, I.I.T, Roorkee for encouragement and also for useful discussions.

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ANNEXURE I

Consider equation (15)

$$\phi = \frac{C}{\sqrt{\psi^2 + \eta^2 + \zeta^2}} = \frac{C}{\sqrt{\frac{x^{*2}}{K_{xx}} + \frac{x^{*2}}{K_{yy}} + \frac{x^{*2}}{K_{zz}}}} \quad \text{(A1.1)}$$

By considering Transverse Isotropic media (T.I), $K_{xx}=K_{yy}= K_1$ and $K_{zz}= K_2$ eqn (A1.1) can be written as

$$\phi_{xx} = \frac{C}{\sqrt{\frac{x^{*2}}{K_1} + \frac{x^{*2}}{K_1} + \frac{x^{*2}}{K_2}}} \quad \text{(A1.2)}$$

By considering eqn (16) now eqn. (17) follows. Darcy's law in eqn. (10) can now be expressed as follows:

$$q^*_x = - K_1 \frac{\partial \phi}{\partial x^*}$$

$$q^*_y = - K_1 \frac{\partial \phi}{\partial y^*} \quad \text{(A1.3)}$$

$$q^*_z = - K_2 \frac{\partial \phi}{\partial z^*}$$

In view of equations (A1.2) and (A1.3) equations (18), (19) and (20) follow.

ANNEXURE II

Consider eqn. (26)

$$Q = CK_1^{3/2} \int_0^{2\Pi} d\phi \int_0^{\Pi/2} \frac{\sin\theta \; d\theta}{1+(\beta^2 -1) \; \cos^2\theta]^{3/2}} \quad \text{(A2.1)}$$

The above integral can be evaluated in two stages.

1. Let us consider inner integral and rewrite it with the following change of variable, by substituting $\cos\theta=t$. Then $-\sin\theta \; d\theta=dt$ and upper and lower limits will be transformed to 0 and 1 and integral can be written as

$$I_1 = \int_0^1 \frac{dt}{[1+(\beta^2 -1) \; \cos^2\theta]^{(3/2)}}$$

$$= \frac{1}{[\beta^2 -1]^{(3/2)}} \int_0^1 \frac{dt}{\left[\frac{1}{[\beta^2 -1]} + t^2\right]^{(3/2)}}$$

$$= \frac{1}{[\beta^2 -1]^{(3/2)}} \left[(\beta^2 -1) \frac{t}{\left[t^2 + \frac{1}{[\beta^2 -1]}\right]}\right]\Bigg|_{t=0}^{t=1}$$

$$= \frac{1}{\beta} \quad \text{(A2.2)}$$

So in view of eqn (A2.2), eqn (A2.1) can be expressed as

$$Q = \frac{CK_1^{(3/2)} \; 2\Pi}{\beta} \quad \text{(A2.3)}$$

Equation (A2.3) is nothing but eqn. (26) in the main text. Hence proved.